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THE AMERICAN MATHEMATICAL MONTHLY

VOLUME XXI

FEBRUARY, 1914

NUMBER 2

THE ALGEBRA OF ABU KAMIL.

By L. C. KARPINSKI, University of Michigan.

Popularity of Algebra Among the Arabs. The great interest in the study of algebra among the Arabs of the Middle Ages is attested by the numerous works written upon this subject by Mohammedan mathematicians. The period of this popularity extends from the ninth to the fifteenth century. The students of algebra included poets, philosophers, and kings. The popular Omar Khayyam was too excellent a mathematician and too fine a poet to woo the muse of algebra in verse. But his treatise in prose on algebra, edited with French translation by Franz Woepcke, is as good a title to fame as his verse. In the library of the Escorial is preserved a *poem* written six hundred years ago by a native of Granada, Mohammed al-Qasim, and treating of algebra. Needless to say the content does not compare with the prose of Omar Khayyam. The King of Saragossa Jusuf al-Mutamin (reigned 1081–1085) was a devoted student of the mathematical sciences. The title of one of his works would seem to indicate its algebraical nature. Even in the study of law a knowledge of algebra seems to have been necessary, for various questions of inheritance were treated by this science. On the whole, however, in that happy day the question of practical applications of mathematics did not loom so large upon the horizon of the Arabic scientist. That fortunate individual felt entirely free to pursue this subject as one of the natural and inevitable activities of the human intellect.

The First Great Arabic Writer on Algebra. The first systematic treatise on algebra was the product of an Arabic writer, Mohammed ibn Musa, al-Khowarizmi (c. 825 A. D.). Commentaries were written upon this work by Arabic authors and two different Latin translations were made of it. One of these translations was published by Libri in his *Histoire des sciences mathématiques en Italie*, Vol. I, 253–297, Paris, 1838. This work is out of print. While the authorship of this version is uncertain the composition may safely be placed in the twelfth century, as several writers of that century appear to have studied this work and use the terminology of this translation. Manuscript copies are somewhat common in

European libraries. French, German, and Italian writers of the fifteenth and sixteenth centuries drew inspiration, and copy, from these pages. The other Latin translation was effected in 1144 A. D. (1183 of the Spanish era) by an Englishman, Robert of Chester, the first translator of the Koran. This version is found, so far as known, only in three manuscripts.¹ The text with English translation and notes is being published in Vol. XI, *University of Michigan Studies, Humanistic Series*. The Arabic text with English translation was published by F. Rosen, London, 1831, but unfortunately this last is out of print. Adequate treatment of this initial work in the science of algebra is found in Cantor's *Vorlesungen über Geschichte der Mathematik*, volume I.

The Second Great Arabic Writer on Algebra. The name and fame of Al-Khowarizmi is known to most students of mathematics. But fate has not dealt so kindly with the second, in point of time, of the great Arabic writers in this field. Abu Kamil Shoja' ben Aslam² wrote in the late ninth or early tenth century a much more extensive treatise on algebra than that of Al-Khowarizmi. Two commentaries of the tenth century have been noted by Arabic historians. In addition to these a commentary in Arabic was prepared by a Spanish Arab, Al-Khoreshi, and a Spanish translation by an unknown Christian Spaniard.³ This was followed by a Hebrew translation, which is preserved in manuscript, by Mordechai Finzi (c. 1473) of Mantua. Not alone by these translations and commentaries was the influence of Abu Kamil exerted upon later writers in this science; for two prominent algebraists made large use of his treatise without, however, mentioning his name. These two men are Al-Karkhi (died c. 1029), whose treatise entitled *Al-fakhri* was analyzed by Woepcke (*Extrait du Fakhri*, Paris, 1853), and Leonard of Pisa (1202). Nor must it be understood that there is any suggestion here of plagiarism, for undoubtedly at the time that Al-Karkhi wrote, Abu Kamil's methods were so well known that any writer could employ his results as common property, while the phraseology of Leonard of Pisa, in the statement of many problems drawn from Abu Kamil, is plainly intended to carry the idea of a citation. Al-Karkhi reproduced not only many problems given by Abu Kamil but also the geometrical solutions devised by this writer, while Leonard of Pisa drew copiously from Abu Kamil for the problems in the section of his *Liber Abbaci* entitled "Expliciunt introductiones algebre et almuchabale; Incipiunt questiones eiusdem."⁴

Abu Kamil's Works. Abu Kamil Shoja' ben Aslam ben Mohammed ben Shoja, the reckoner from Egypt, was an excellent and learned arithmetician. He wrote: The book of fortune, The book of the key to fortune, The book on algebra, The book of extracts, The book of omens (by the flight of birds), On

¹ Karpinski, "Robert of Chester's Translation of the Algebra of Al-Khowarizmi," *Bibliotheca mathematica*, third series, XI, 125-131.

² Karpinski, "The Algebra of Abu Kamil Shoja'ben Aslam," *Bibliotheca mathematica*, third series, XII (1912), 40-55. To this article the reader is referred for more complete bibliographical references and for citations from the Latin text.

³ H. Suter, in a personal communication to the author.

⁴ *Scritti di Leonardo Pisano*, published by Prince Boncompagni, Rome, 1857, I, 410-459.

completion and diminution, The book on the rule of double false position, The book of surveying and geometry, The book of the adequate (possibly in arithmetic).¹ Such is the account of the life and works of Abu Kamil as given in the *Kitab al-Fihrist* (987 A. D.) of An-Nadim, who includes this writer in his list of the later (*i. e.*, nearly contemporary with An-Nadim) reckoners and arithmeticians. Aside from the works included in this list a treatise on the mensuration of the pentagon and the decagon exists in Latin and Hebrew translations, and a further arithmetical work is preserved in the original Arabic as well as in Latin and Hebrew. The work on the pentagon and decagon has been published in Italian by Sacerdote,² and in German by Suter.³ The arithmetical work, in which he deals with first degree indeterminate equations, has been translated by Suter.⁴ Other information about the life and activity of this great Arab we do not have. The titles mentioned indicate that, like so many of the later scientists in Europe, Abu Kamil was interested in magic, doubtless including astrology, and in omens.

The Sources of Information. The manuscript upon which I have based my study of Abu Kamil's algebra is the Paris manuscript 7377A of the Bibliothèque nationale.⁵ The treatise covers folios 71^v–93^v, and contains between 35,000 and 50,000 words.⁶

In the opening sentence the author refers to his famous predecessor in this field, Mohammed ibn Musa al-Khowarizmi, and a little later again refers to the algebra of the same writer. Several references are made to Euclid but aside from these two names no others have been noted in this algebra.

Quotations from the Manuscript. The following translation of selected passages from the algebra is a very free one, as an attempt is made to preserve the mathematical significance. Additions to clarify the meaning, and modern notations, are put in parentheses.

The first thing which is necessary for students of this science is to understand the three species which are noted by Mohammed ibn Musa al-Khowarizmi in his book. These are *roots*, *squares* and *number*. A root is anything which can be multiplied by itself, composed of one, and numbers above one, and fractions. A square is that which results from the multiplication by itself of root, composed of units, and units and fractions. A number is a quantity by itself which does not have the name of root or square, but is proportional to the number of units which it contains. These three species are proportional to each other by turn in twos. Thus, squares equal to roots, squares equal to number, and roots equal to number ($ax^2 = bx$, $ax^2 = n$, $bx = n$).

An illustration of squares equal to roots is this:

A square is equal to five of its roots ($x^2 = 5x$). The explanation of this is that a square is five times the root of it. The root of the square is always the same as the number of the roots to which the square is equal. In this question this root is five and the square is twenty-five

¹ I translate from the German translation of the passages in the *Fihrist* dealing with mathematical scientists, by Suter, in *Abhandl. z. Geschichte d. math. Wissen.*, VI, 1–87.

² *Festschrift Steinschneiders*, Leipzig, 1896, 169–194.

³ *Bibliotheca mathematica*, X, third series, 15–42.

⁴ *Das Buch der Seltenheiten der Rechenkunst*, in *Bibl. math.*, 3d ser., XI, 100–120.

⁵ I am indebted to the librarian for permission to have photographic reproductions made.

⁶ To give an idea of the extent of this work, if the Latin text were printed in the MONTHLY it would take between 75 and 100 pages.

which is the same as five of its roots. That the root of the square is the same as the number of the root this I will explain.

We place for the square a square surface $abgd$ whose sides are ab , bg , gd , and da (see Fig. 1). Of this square any side multiplied by one in number is the same as the area of a surface whose length is the side or root of the square. The side ab multiplied by one which is be gives the surface ae which is the root of the square ag . The surface ag is the same as five roots or five times the root of itself. Therefore, it is five times the surface ae . Divide then, the surface ag by equidistant lines into five equal parts, that is the surfaces ae , ck , sr , nh , and mg . The lines be , ek , kr , rh and hg are equal. Moreover the line be is one. Therefore the line bg will be five which is the root of the square and the square itself is twenty-five which is the surface ag . This is what we wished to explain.

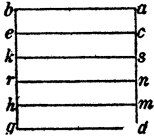


FIG. 1.

If it is proposed that one half a square should equal 10 roots ($\frac{1}{2}x^2 = 10x$) then the whole square is 20 roots. The root of the square is 20 and the square 400. Similarly if it is proposed that 5 squares equal 20 roots ($5x^2 = 20x$) then the root of the square is 4 and the square is 16. And whether more than a square or less than a square (is proposed) you reduce to 1 square. You operate in the same manner with that which is connected with the square, the roots.

A square which is equal to a number is illustrated:

A square equals 16 ($x^2 = 16$). Therefore, it itself is 16 and the root of it is 4. Similarly if given 5 squares equal 45 drachmas ($5x^2 = 45$), then 1 square is a fifth of 45 which is 9 and the root of it is 3. . . . So whether more or less than a square (is given) it is reduced to 1 square and similarly is treated that which is joined or equal to them of numbers.

Roots which are equal to numbers are illustrated:

A root equals 4 ($x = 4$). Therefore, the root itself is 4 and the square is 16. Similarly again if it is proposed that 5 roots should equal 30 ($5x = 30$) then the root is equal to 6 and the square 36. . . .

We construct now these three species, that is roots, squares and numbers, joined by twos and proportional to the third. This gives, a square and roots equal to number, a square and number equal to roots, and roots and number equal to squares.

(In modern notation $ax^2 + bx = n$, $ax^2 + n = bx$, $bx + n = ax^2$.)

First Type of Quadratic Equation. An illustration of squares and roots equal to number is the following:

A square and 10 roots equal to 39 drachmas ($x^2 + 10x = 39$). The explanation of this is that there is some square to which if you add the same as 10 of its roots the number of it results in 39 drachmas. In this question there are two ways (methods) of which one leads you to know the root of the square and the other to know the square of the root. We will set forth and explain each of these by a geometrical figure which will be understood by those who understand the book of Euclid, that is the Elements. The method which leads us to know the root of the square has been narrated by Mohammed ibn Musa al-Khowarizmi in his book. This is that you divide the roots in half, that is in two equal parts. In this question this gives 5. You multiply this by itself, giving 25, which you add to 39. This gives 64 of which number you take the root which is 8; from this you take the half of the roots, that is 5, and there remains 3, which is the root of the square. The square is 9.

The method indeed which leads you to know the square is that you multiply 10 roots by itself, giving 100. This you multiply by 39, which is equal to the square and roots, giving 3,900. Now divide 100 in halves, giving 50, which you multiply by itself, giving 2,500. This add to 3,900, giving 6,400, and of this number take the root, which is 80. This you subtract from 50, which is the half of 100, and from 39, which is equal to the square and the roots, together giving 89, and 9 remains, which is the square.¹

¹ The text to this point is found in the Paris MS. on folio 71^v-72^a; given in my article in the *Bibliotheca Mathematica*, loc. cit., p. 42-44.

In modern notation the steps of this solution are indicated as follows, in which both the general case and this special problem are given in parallel columns.

$$\begin{array}{ll}
 (1) & x^2 + bx = n, & x^2 + 10x = 39, \\
 (2) & b^2x^2 + b^2x = nb^2, & 100x^2 + 1000x = 3,900, \\
 (3) & b^2x^2 + b^2x + \left(\frac{b^2}{2}\right)^2 = nb^2 + \frac{b^4}{4}, & 100x^2 + 1000x + 2500 = 6,400, \\
 (4) & bx + \frac{b^2}{2} = +\sqrt{nb^2 + \frac{b^4}{4}}, & 10x + 50 = 80.
 \end{array}$$

Only the positive root is taken, as the negative root leads to the negative solution of the equation. Subtracting (4) from (1), member for member,

$$\begin{aligned}
 x^2 - \frac{b^2}{2} &= n - \sqrt{nb^2 + \frac{b^4}{4}}, & x^2 - 50 &= 39 - 80, \\
 x^2 &= n + \frac{b^2}{2} - \sqrt{nb^2 + \frac{b^4}{4}}, & x^2 &= 50 + 39 - 80, \\
 & & x^2 &= 9.
 \end{aligned}$$

Following this the problem is proposed:

$$2x^2 + 10x = 48.$$

This is a problem which is also given by Al-Khowarizmi and the same is true of many further problems given by Abu Kamil. The latter author gives the method of solution presented by Al-Khowarizmi and also the method, above illustrated, leading to the value of x^2 directly. Further than this he presents the geometrical solution for the first method as given by his more famous predecessor. Abu Kamil adds the geometrical solution corresponding to his own method, the second of those given above. This seems to be worth presenting here.

In order to show you the method (the door, literally) which leads to a knowledge of the square we place for the square the line ab (see Fig. 2). To this we add 10 roots of itself which are represented by the line bg , giving the line ag , which is 39. We wish to know then the value of the line ab . We construct then upon the line bg a square surface, which is the surface $degb$. This then will be $100x^2$ (in modern notation), that is, 100 times the line ab We construct then the surface ah the same as the square surface be , that is equal to the square, which is the same as the multiplication of the line ab in one of the units which it contains taken 100 times. Then we complete the surface an which is 3,900, since the line ag is 39 and am is 100, which lines contain this surface. Moreover the surface ah is the same as the surface be . Therefore, the surface dn is 3,900, which is produced by the multiplication of the line ne by eg , as the line eg is the same as the line ed and the line gn is 100 since it is the same as the line am . Therefore we divide the line gn in two equal parts at the point l , to which line so divided the line ge is added in length. Therefore, the multiplication of the whole line ne by eg together with the product of the line gl by itself is the same as the product of the line le by itself, as Euclid says in the second book¹ of his *Elements*. The product of ne by eg is 3,900 and the product of gl

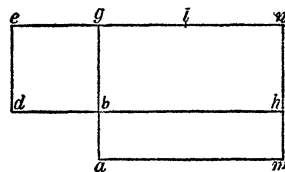


FIG. 2.

¹ Euclid's *Elements*, II, 6, "If a straight line is bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line, together with the square on the half, is equal to the square on the straight line made up of the half and the added straight line."

by itself is 2,500, which being combined make 6,400, which is the same as the product of the line le by itself. Therefore, the product of the line le by itself is 6,400, of which the root is 80, which is the line le . But the line ge is the same as the line gb , and so the line lg and the line gb make 80. When we take away the line lg and the line dg , which are 80, from the lines ag , 39, and gl , 50, which make 89, the line ab remains, which is 9. This is the square and that is what we desired to explain.¹

Second Type of Quadratic Equations. The first problem which Abu Kamil presents of the second type of composite quadratic equations is again the same as that given by Al-Khowarizmi,

$$x^2 + 21 = 10x.$$

For this type of equation also Abu Kamil presents two solutions, the one leading directly to the value of the root of the equation and the other to the value of the square of the root. Further the author notes that there are two solutions for both the root and the square in this type of quadratic equation. The reason that this was noted by both of these Arabic writers is that in this type of equation the two roots are positive, whereas in the other two types the one root is positive and the other root is negative. Negative quantities as such are not accepted by these Arabic mathematicians and this, of course, accounts for the three types of quadratic equations. Express mention is also made of the fact that if the square of half the coefficient of the roots is less in magnitude than the constant then the problem is impossible, from the Arabic standpoint, or contradictory. Further also, the author mentions that when this square is equal to the given constant the root of the equation is the same as one half the given coefficient.

In the above problem,

$$x^2 + 21 = 10x,$$

the analytical solution leading to the value of the square of the unknown quantity proceeds in substance as follows:

Multiply 10 by 10, giving 100. Multiply this by 21, giving 2,100. Then take half of 100, giving 50, which you multiply by itself, giving 2,500. From this subtract 2,100, leaving 400 of which the root is 20, which you subtract from 50, the half of 100, leaving 30. From this you subtract 21, leaving 9, which is the square. And if you wish, add 20 to 50, giving 70, from which you subtract 21, leaving 49, which is the square and the root of it is 7.²

Al-Khowarizmi gave the geometrical figure corresponding only to the smaller root of the equation. Thus in the equation, $x^2 + n = bx$, in which

$$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - n},$$

one of these roots, when both are real, is necessarily less than $b/2$ and the other is greater. Abu Kamil gives the geometrical explanation corresponding to both, thus completing the work of Al-Khowarizmi. Thus he says that when you assume that the square is less (in numerical value) than the number which is given, then the solution appears by subtraction, and when you assume the same

¹ In the manuscript folio 72^v-73^r; in *Bib. Math., loc. cit.*, p. 46.

² Manuscript folio, 73^r; *Bib. Math., loc. cit.*, p. 47.

greater than the constant the solution appears by addition. Further he gives a geometrical solution for the type in which the square of the coefficient of x is equal to the given constant. The figure in this case consists simply of two equal squares with a side in common. We follow with a demonstration, original with Abu Kamil, leading to value of x^2 in the equation, $x^2 + 21 = 10x$.

... Place the line ab to represent x^2 (Fig. 3), and we add to it the drachmas which accompany it, 21, and let it be the line bg . Therefore, the line ag is 10 roots of the line ab . We construct then upon the line ag , a square surface, the square $aged$ and this is 100 times the line ab ... since the line ag is 10 roots of the line ab , and 10 roots multiplied by itself gives 100 squares (x^2). We construct then the surface ah equal to the square $aged$ and let one side of it be the line ab . Therefore, the other side bh is 100. Then we complete the surface an and therefore the surface bn is 2,100 since the line bg is 21 and the line gn 100. We construct then the surface my equal to the surface ae , but the surface ae is equal to the surface ah . Take away, then, the surface th and there remains the area cn equal to the area ca . Adding then the surface by , the whole surface ay is equal to the surface bn , which is 2,100. Therefore the area ay is 2,100 which is the product of gy by yn , since yn is equal to yt , since the area tn is a square. We divide then the line gn in two equal parts at the point l . Therefore the line ng is divided into two equal parts by the point l and into two unequal parts by the point y . Therefore the product of gy by yn together with the square upon ly is the same as the product of ln by itself.¹ But ln by itself makes 2,500, since it is 50, and the product of gy by yn is 2,100. The square on the line ly is then left as 400 and the line ly 20. But ln is 50 and as it (ly) is to be subtracted the line yn is left as 30. Since the line bg is 21 the remaining line ab is 9 which is the square. This is what we wished to explain.²

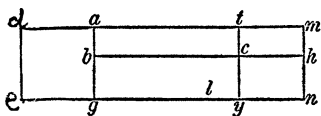


FIG. 3.

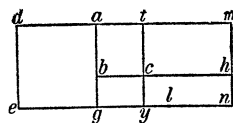


FIG. 4.

Having given the explanation "by subtraction," that is to say, when the square root of $2,500 - 2,100$ is subtracted from 50, Abu Kamil proceeds to explain "by addition." gn is again divided into two equal parts but the point of division falls between y and n (Fig. 4). This corresponds to the assumption that the square is greater than the number which accompanies it. This has also been noted above.

The product of gy by yn together with the square of yl equals the product of lg by itself. lg multiplied by itself gives 2,500 and the line yn by yg gives 2,100. Therefore the square of ly is 400 and ly is 20. Since ln is 50, the whole line yn is 70. But yn is the same as yt and yt equals ag . Therefore ag is 70. But bg is 21 whence ab is 49 which is the square. This is what we desired to explain.³

In the third type of quadratic equations Abu Kamil again follows Al-Khowarizmi in taking the equation $3x + 4 = x^2$. Three explanations are presented for finding the root of the square. The first is equivalent to the demonstration presented by Al-Khowarizmi but the figure is not completed. Instead of this the proof is made to depend upon the second book of Euclid. The second gives

¹This is by Euclid VI, 5, "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with square on the straight line between the points of section, is equal to the square on the half."

²Manuscript folio, 74^r; *Bib. Math.*, loc. cit., pp. 48-49.

³Manuscript folio, 74^r; *Bib. Math.*, loc. cit., p. 49.

the completed figure and explains as in the algebra of Al-Khowarizmi. In his third figure (Fig. 5) the square $abdg$ is placed to represent x^2 which by the conditions of the problem equals $3x + 4$. Then the lines ah and yd are taken $1\frac{1}{2}$ units in length and the rectangles completed cutting off from the original square $3x$ less the little square, $1\frac{1}{2}$ on a side, which these two rectangles have in common. Hence the square which remains $hnyz$ is $4 + (1\frac{1}{2})^2$ or $6\frac{1}{4}$. It follows that gh is $2\frac{1}{2}$ and ay (or x) is 4.

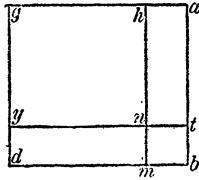


FIG. 5.

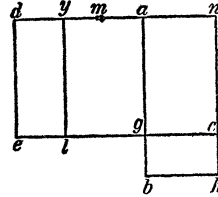


FIG. 6.

To arrive at the value of x^2 geometrically Abu Kamil places (Fig. 6) ab for the square and cuts off gb equal to 4, leaving ag equal to the three roots. Then he continues:

Upon ag a square is constructed $aged$ whose area is $9x^2$ or 9 times the line ab . ah is constructed equal to the square ae . From this it follows that an is 9 and that gc which equals an is 9. Since bg is 4 the area of gh is 36. Take ay equal to an and draw yl parallel to de . The area yg equals ac and the area ae equals ah . Hence ye equals cb which is 36. But ya is 9 since an is 9. Divide ay into two equal parts by the point m . Now yd is added in length to ay . Hence ad by dy plus ym multiplied by itself equals md multiplied by itself, as EUCLID says in his second book. But ad by dy is ye or 36. ym multiplied by itself is $20\frac{1}{4}$. Adding you obtain $56\frac{1}{4}$. Therefore md multiplied by itself gives $56\frac{1}{4}$ and md is $7\frac{1}{2}$. But am is $4\frac{1}{2}$ and gb is 4. Hence ab is 16 and this is what we desired to demonstrate.¹

Abu Kamil summarizes the results of the algebra up to this point, stating that all questions that can be solved by *algebra and almuchabala*, that is, restoration and opposition, must reduce to one or the other of the six types of equations mentioned.

Multiplication and Division. In the next section the multiplication and division of algebraic quantities is considered. Binomials, involving both addition and subtraction, are given particular consideration. The writer uses *res* and *radix*, indifferently, for the first power of the unknown. This is combined by addition or subtraction with a pure number. The second power of the unknown is designated by *census*, but this word is sometimes used simply for any unknown quantity as in problems below. Similar usage obtains in the work of Leonard of Pisa and in the *Liber augmenti et diminutionis* by one Abraham, published by Libri, *Histoire*, I, 304-371.

After a general statement about multiplication which summarizes the work presented, Abu Kamil continues with a geometrical explanation of the problems. Thus a figure containing four small squares is used to illustrate the fact that $2x$ multiplied by $2x$ gives $4x^2$. The next problem is $3x$ multiplied by 6 drachmas.

¹ Manuscript folio, 75r; *Bib. Math., loc. cit.*, pp. 49-50.

In this section the constant is referred to drachmas although in other places the word *numerus* is also used. Figures are shown for $10 + x$ by x , $10 - x$ by x , $10 + x$ by $10 - x$ and other similar examples.

Radical Expressions. The addition and subtraction of radicals, involving quadratic irrationalities only, is effected by means of the relations given in modern symbols by the equalities $\sqrt{a} \pm \sqrt{b} = \sqrt{a \pm b \pm 2\sqrt{ab}}$. Thus, to subtract the square root of 8 from the square root of 18 the rule is simply: "subtract 24 from 26, as you know, and 2 remains. The root of this is the root of 8 subtracted from the root of 18." This problem, by the way, is given by Al-Karkhi. Leonard of Pisa¹ employs the same method but uses 18 and 32, instead of 8 and 18, combining them by addition and subtraction in the same way as Abu Kamil.

Al-Karkhi² proceeds to the consideration of cube roots in a similar way but our author takes up for a final problem in roots the addition of $\sqrt{10}$ to the $\sqrt{2}$ which, he states, does not lead to a simple result since $10:2 = 5$ which is not a square and also $2:10 = \frac{1}{5}$ which is not a square. This section concludes with the statements that $\sqrt{10} + \sqrt{2} = \sqrt{12 + 2\sqrt{20}}$ and $\sqrt{10} - \sqrt{2} = \sqrt{12 - 2\sqrt{20}}$.

Problems. The section dealing with problems is introduced in much the same way that Al-Khowarizmi does the corresponding section, and many of the problems are taken from the older work. The first problem is to divide ten into two parts such that the square of the larger part should equal $1\frac{1}{2}$ times the product of the two parts. This leads to the equation

$$x^2 = 1\frac{1}{2}x(10 - x),$$

or

$$2\frac{1}{2}x^2 = 15x.$$

Thus the problem leads to the first of the six types of quadratic equations.

The above problem is the second presented by Leonard of Pisa³ in the section with the heading: *Expliciunt introductiones algebre et almuchabale. Incipiunt questiones eiusdem*. Furthermore this problem does not appear in Al-Karkhi⁴ in the list given by Woepcke. The conclusion is reasonable that Leonard had some source of information concerning the algebra of Abu Kamil other than Al-Karkhi's work, and even more particularly in view of the long series of problems found in Leonard's work which correspond to those given by Abu Kamil. The second and third questions here are found to be the same as the following two questions in Leonard of Pisa,⁵ but with slightly different numbers. Thus instead of $6\frac{1}{4}x^2 = 100$, Leonard and Al-Khowarizmi give $2\frac{7}{9}x^2 = 100$ and in place of $x/(10 - x) = 4$ as given by Abu Kamil and Al-Khowarizmi, Leonard sets this same fraction equal to $2\frac{1}{3}$. Our space does not permit an examination of all similarities of this kind.

¹ *Scritti di Leonardo Pisano*, published by Boncompagni, Vol. I: *Il liber abbaci*, Roma, 1857, p. 363-365.

² F. Woepcke, *Extrait du Fakhri*, Paris, 1853, p. 57-59.

³ *Liber abbaci*, p. 410.

⁴ Woepcke, loc. cit., p. 75-137.

⁵ *Liber abbaci*, p. 410.

A problem given on folio 81^r is of particular interest because of its similarity to a problem of Al-Khowarizmi's¹ which has given rise to discussion. The problem is to divide 50 among some number of men and then to add three men and again distribute 50 (drachmas) among them (equally). In the second instance each man receives $3\frac{1}{2} \frac{1}{4}$ (meaning $3 \frac{1}{2} + \frac{1}{4}$) drachmas less than in the first. Leonard states the corresponding problem as follows:

Distribute 60 among (a certain number of) men and each will receive something. Add two men and among them all again distribute 60 and then each man will receive $2\frac{1}{2}$ denars less than at first.

The statement of Abu Kamil is:

And if we tell you, distribute 50 among (a certain number of) men and each receives a certain amount (*res*). Add three men and distribute again 50 drachmas among them. Each then receives $3\frac{1}{2} \frac{1}{4}$ drachmas less than before.

Both follow with a geometrical explanation.

Problems Involving Fractions. The notation of fractions employed by this translator of Abu Kamil resembles the peculiar system employed later by Leonard. Thus on fol. 83^r the square of $\frac{1}{2}$ and of $\frac{1}{9}$ are given in combinations as in the following translation:

... $\frac{1}{2}$ and $\frac{1}{9}$ which multiplied by itself, gives $\frac{1}{4}$ and $\frac{1}{9}$. To this we add 11 drachmas and $\frac{1}{9}$, giving 11 drachmas and $\frac{1}{4}$ and $\frac{1}{6}$ and $\frac{e}{9} \frac{1}{2}$ and $\frac{e}{9} \frac{1}{9}$. Of this we take the root or 3 and $\frac{1}{3}$ and $\frac{0}{9} \frac{1}{2}$...

The fraction $\frac{1}{9} \frac{1}{9}$ stands for $\frac{1}{9}$ plus $\frac{1}{81}$ whereas $\frac{e}{9} \frac{1}{2}$ as well as $\frac{0}{9} \frac{1}{2}$ represents simply $\frac{1}{18}$. So also Leonard² uses $\frac{5}{8} \frac{e}{8}$ for $\frac{5}{64}$. This system is explained by Leonard³ who writes for $\frac{1}{14}$ the form $\frac{1}{2} \frac{0}{7}$ and $\frac{1}{2} \frac{5}{6} \frac{7}{10}$ for $\frac{7}{10} + \frac{5}{60} + \frac{1}{120}$. The difference is only in the order of reading.

Al-Khowarizmi gives several problems of this nature:⁴

To find a square of which if one-third be added to three dirhems, and the sum be subtracted from the square, the remainder, multiplied by itself restores the square.

The solution involves the use of x (root) to represent the square. So also Leonard⁵ in a similar problem states: "... place for that square x (*res*).". Our version of Abu Kamil has several problems of the same kind:

And if we tell you there is a quantity (*census*) from which when you subtract $\frac{1}{4}$ of itself and 2 drachmas, the product of the remainder by itself gives the quantity and 24 drachmas.

The solution is obtained by placing x (*res*) for the quantity (*census*).

¹ Libri, *Histoire des sciences mathématiques en Italie*, Vol. 1, Paris, 1838: "Liber Maumeti ... de algebra et almuchabala," p. 286. Rosen, *The Algebra of Mohammed ben Musa*, London, 1831, p. 63-64. Al-Khowarizmi does not present any geometrical explanation.

² *Liber abbaci*, p. 447.

³ *Liber abbaci*, p. 24, 25.

⁴ Rosen, *loc. cit.*, p. 56, 57.

⁵ *Liber abbaci*, p. 422.

The longest discussion of any algebra problem in Leonard is that of the following:¹

Divide 10 in two parts, and divide the larger by the smaller, and the smaller by the larger. Sum the results of the division and this equals the square root of 5.

Leonard presents several solutions. In the first he arrives at the equation $\sqrt{5x^4 + 2x^2 + 100} = 20x + \sqrt{500x^2}$. The coefficient of x^2 is made equal to unity by multiplying through by $\sqrt{5} - 2$. The same equation is found in Abu Kamil and the treatment is the same. The value of x is $5 - \sqrt{225 - \sqrt{50000}}$ which is found by both writers. Leonard goes on to find the value of $\sqrt{225 - \sqrt{50000}}$ as $\sqrt{125} - 10$. Al-Khowarizmi and Al-Karkhi do not give this problem although they do give one upon which this is based, namely: to divide 10 into two parts such that the sum of each divided by the other is $2\frac{1}{6}$. Al-Karkhi² discusses four solutions.

Leonard of Pisa occasionally solves incorrectly, if this expression be permissible. Thus the problem:³ To find a number such that, if the square root of 3 be added to it and then the square root of 2 be added, the product of the two sums will be 20. Algebraically $(x + \sqrt{3})(x + \sqrt{2}) = 20$. The product of the two binomials is given as $x^2 + 6 + \sqrt{12x^2} + \sqrt{8x^2}$. This leads to the incorrect value $\sqrt{19 + \sqrt{24}} - \sqrt{3} - \sqrt{2}$ for x . Our manuscript of Abu Kamil gives the correct value, $\sqrt{21\frac{1}{4} + \sqrt{1\frac{1}{2}}} - \sqrt{6} - \sqrt{\frac{3}{4}} - \sqrt{\frac{1}{2}}$, for x . The problems⁴ following this in the algebra of Leonard, with the exception of the two final ones, are all taken from Abu Kamil and in the same order in which they occur in the work of the Arabic writer.

The Italian writer frequently uses an expression which might be supposed to refer to Abu Kamil: "Operate according to algebra,"⁵ . . . "Operate therefore in this according to algebra, etc." However our manuscript employs a similar expression. Thus in a problem arriving at $16x^4 = 256x^2$, this Latin version of Abu Kamil reads: *Fac secundum algebra in eis et erunt 16 census census equales 256 censibus. census census (x¹) ergo equatur 16 censibus et census equatur 16 dragmis, ergo res equatur 4 dragmis . . .* (fol. 91^r). In the discussion of the same problem Leonard concludes one form of solution with the words: "Age ergo in eis secundum algebra, et inuenies, census census equari 16 censibus: quare census est 16, et radix eius est 4, ut dictum est." Both writers present several solutions, including a geometrical one, of the problem in question which is to divide 10 into two parts such that if from the larger you subtract two of its roots and to the smaller add two of its roots the quantities are then equal. The expression *secundum algebra* refers to the Book of Algebra and Almucabala by Mohammed ben Musa.

¹ *Liber abbaci*, p. 434-438.

² Woeppke, *loc. cit.*, p. 91, 92.

³ *Liber abbaci*, p. 445.

⁴ *Liber abbaci*, 445-459.

⁵ *Liber abbaci*, 438.

Treatise on the Pentagon and Decagon. The algebra terminates with the first eight lines on fol. 93^v, and is followed by the work on the pentagon and decagon by the same author. Although this treatise on the pentagon and decagon by Abu Kamil is geometrical in its nature yet the treatment and the solutions are algebraical, including a fourth degree equation ($x^4 = 8000x^2 - \sqrt{51\,200\,000}$) as well as mixed quadratics with irrational coefficients. The twelfth problem is in the Latin: "Et si dicemus tibi trianguli equilateri et equianguli mensura est cum perpendiculari ipsius est 10 ex numero, quanta sit perpendicularis?"¹ This suggests the similar problems in Greek of unknown date and author presented by Heiberg and Zeuthen,² as well as the similar problems given by Diophantos. In these Greek problems also lines and areas are summed quite contrary to ancient Greek usage. A further point of interest is that in the equation $x^2 + 75 = 75x$, to which the solution of the thirteenth problem leads, Abu Kamil gives only one solution whereas in the algebra he recognizes that this equation has two positive roots. The opening sentence of this treatise on the pentagon and decagon makes reference to the algebra as immediately preceding it, which indeed is the fact in the Hebrew and Latin manuscripts that are preserved.

Conclusion. The algebra terminates with a general statement to the effect that by the methods taught in this book many more problems can be easily solved. In true Arabic fashion, the closing words are: "Whence praise and glory be to the only Creator."

Let us summarize the results of our study. The most important conclusion of this investigation of Abu Kamil's algebra is that Al-Karkhi and Leonard of Pisa drew extensively from this Arabic writer. Through them this man, though himself comparatively unknown to modern writers, exerted a powerful influence on the early development of algebra. Abu Kamil deserves somewhat the same recognition from modern mathematicians and historians of science as that which Leonard of Pisa and Al-Khowarizmi have received. We may hope that the future will be more just than the past in according to Abu Kamil a prominent place among the mathematicians of the middle ages.

A CURIOUS CONVERGENT SERIES.

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It is well known that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. The object of this Note is to prove that if the denominators do not

¹"If we say to you that an equilateral and equiangular triangle, together with its altitude, is measured by 10, what is the altitude?"

²*Einige griechische Aufgaben der unbestimmten Analytik*, in *Biblioth. mathem.*, VIII, third series, 118-134.